Chapter 5: Proofs

Example 5.1

Prove that the difference of any two odd integers is an even integer.

Example 5.2

Which of these are statements?

(a) The trains are always late.

(b) 12 is a prime number.

(c) $7 + 8 = 15$

(d) Welcome to the USA.

(e) $7 + 8 = 16$

(f) $n + 1 \leq 15$

Example 5.3

Find the negation of each of the following statements and write them in simple English:

(a) P: Kevin is looking for a new car.

(b) Q: 152 is a perfect square.

(c) R: Denny walked his dogs 5 km on Sunday.

Example 5.4

Find the conjunction and disjunction of the statements M and N, where M is the statement 'Elvin's PC has more than 100 GB free hard disk space', and N is the statement 'The processor in Elvin 's PC runs faster than 2 GHz'.

Example 5.5

Suppose you tell your younger brother 'If the concert is sold out, I will take you to the movies.' This promise is broken (the conditional sentence is false) only when the concert was sold out (the antecedent is true) and you did not take your brother to the movies (the consequent is false). This is row 2 of the truth table. In all other situations, the promise is true. If there were tickets left (row 3 and 4 of the table), we don't say the promise was broken, regardless of whether you decided to go to the movies. The promise is also kept in the situation where the concert is sold out and you went to the movies, which is row 1 of the table.

Example 5.6

Write both the converse and the contrapositive for each statement below. In each case, determine whether the given statement is true or false, state whether the converse is true or false, and state whether the contrapositive is true or false.

(a) If $7 < \sqrt{50}$, then 8 is a composite number.

(b) If $7 \geq \sqrt{50}$, then 8 is a composite number.

(c) If $7 < \sqrt{50}$, then 8 is a prime number.

(d) If $7 \geq \sqrt{50}$, then 8 is a prime number.

Example 5.7

Suppose that the conditional statement 'If you score more than 80% on your final exam, you will receive a grade 7 for the whole term' and its hypothesis, 'You scored more than 80% on your final exam', are true. Then, by modus ponens, it follows that the conclusion of the conditional statement, 'You will receive a grade 7 for the whole term', is true.

Example 5.8

Show that the product of two odd integers is an odd integer.

Example 5.9

Let $m, n \in \mathbb{Z}$. Prove that m and n are both odd if and only if (iff) the product mn is odd.

Example 5.10

Prove the following statement: If $m, n \in \mathbb{Z}$, and m is divisible by n, then m is divisible by kn, where k is any integer.

Example 5.11

Prove the following statement: If $m, n \in \mathbb{Z}$, and m is divisible by n, then m^2 is divisible by n^2.

Example 5.12

Prove the following statement: If $n \in \mathbb{Z}$ and n is not a multiple of 5, then n^2 has only remainders 1 or 4 upon dividing by 5.

Example 5.13

Consider the real numbers m and n. Prove that if m is rational and n is irrational, then $m + n$ is irrational.

Example 5.14

Prove that $\sqrt{2}$ is irrational.

Example 5.15

Prove that for $n \in \mathbb{Z}$, if $3n + 2$ is odd, then n is odd.

Example 5.16

Prove that for any real number x, if x^2 is irrational, then x is irrational.

Example 5.17

Prove the statement from example 5.11: m and n are both odd if and only if the product mn is odd.

Example 5.18

Prove that for any two integers m and n, if $m + n$ is odd, then exactly one of m or n is odd.

Example 5.19

Prove that for all integers a, b, and c, if $c \nmid ab$ then $c \nmid a$ and $c \nmid b$.

Example 5.20

Prove that for $x \in \mathbb{Z}$, if $x^2 + 6x - 5$ is even, then x is odd.

Example 5.21

Prove that for a positive real number a, if a is irrational then $\sqrt{a}$ is irrational.

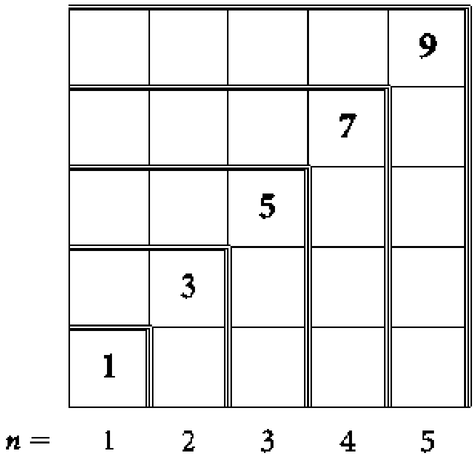
Example 5.22

Prove that the sum of the first n natural numbers is given by this formula:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Example 5.23

In an investigation to find the sum of the first n positive odd integers, we can do the following: Investigate the sums of the first few odd integers and then try to come up with a conjecture. Then mathematical induction will provide us with a tool to prove the conjecture.



For $n = 1$, the sum is $1 = 1$

For $n = 2$, the sum is $1 + 3 = 4$

For $n = 3$, the sum is $1 + 3 + 5 = 9$

For $n = 4$, the sum is $1 + 3 + 5 + 7 = 16$

For $n = 5$, the sum is $1 + 3 + 5 + 7 + 9 = 25$

When we compare the number of integers being added with their sum, they are related; that is, the sum of n such integers is n^2.

n	1	2	3	4	5	6	...	n
sum	1	4	9	16	25	36	...	n^2

Example 5.24

Prove that $3^n < n!$ for all integers $n > 6$.

Example 5.25

Show that in an arithmetic sequence where $a_n = a_{n-1} + d$, the nth term can be given by the formula:

$$a_n = a_1 + (n-1)d$$

Example 5.26

Show that in an arithmetic series: $S_n = \frac{n}{2}(2a_1 + (n-1)d)$

Example 5.27

Show that 3 divides $n^3 + 3n$ for all non-negative integers n.

Example 5.28

Using mathematical induction, show that for all non-negative integers n:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$