Chapter 16: More integral calculus

Example 16.1

Verify that $y(x) = Ce^{x^4}$ is a solution to the differential equation

$$\frac{dy}{dx} = 4x^3y$$

Example 16.2

Verify that $y(x) = -\dfrac{1}{2x^4 + 3}$ is a solution to the differential equation

$$\frac{dy}{dx} = 8x^3y^2$$

Example 16.3

Find a solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{x+2}, \quad x \neq -2$$

Example 16.4

Find the general solution of the differential equation

$$y' - 9x^2y^2 = 5y^2$$

Example 16.5

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x^2y}{1 + 4y^2}$$

Example 16.6

Find the general solution of the differential equation

$$x^2y\frac{\mathrm{d}y}{\mathrm{d}x} = x + 1, \quad x > 0, \quad y > 0$$

Example 16.7

Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x + 1};\ y(1) = 8$$

Example 16.8

Solve the initial value problem

$$\frac{\mathrm{dy}}{\mathrm{d}t} = (e^{y-t})\frac{1+t^2}{\cos y};\ y(0) = 0$$

Example 16.9

Solve the differential equation $2x\mathrm{d}x + e^{x+y}\cos y\,\mathrm{d}y = 0$

Example 16.10

Find the general solution of the population growth model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

Example 16.11

A cold object is placed in warmer medium that is kept at a constant temperature S. The rate of change of the temperature $T(t)$ with respect to time is proportional to the difference between the surrounding medium and the object and hence it satisfies

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(S - T) \text{ and } T(0) = T_0$$

where $k > 0$ and $T_0 < S$; that is, the initial temperature is less than the temperature of the surrounding medium. Find the solution to the initial value problem.

Example 16.12

The rate of decay of a substance y at any time t is directly proportional to the amount of y and also directly proportional to the amount of another substance x.

The constant of proportionality is $-\frac{1}{2}$ and the value of x at any time t is given by $x = \frac{4}{(1+t)^2}$

(a) Given the initial conditions that $y = 10$ when $t = 0$, find y as an explicit function of t.

(b) Determine the amount of the substance remaining as t becomes very large.

Example 16.13

Find the general solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$

Example 16.14

Solve the logistic differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{L}\right)$

Example 16.15

An official wildlife commission releases 20 deer into a protected forest.
After 10 years, the deer population is 208. The commission believes that the environment cannot support more than 400 deer.

(a) Write a model for the deer population in terms of the number of years, t.

(b) Use the model to estimate the deer population after 10 years.

(c) Discuss the long-term trend of the deer population.

Example 16.16

Solve the differential equation $xyy' = x^2 + y^2$

Example 16.17

Find the general solution of $(x^2 - y^2)\mathrm{d}x + xy\mathrm{d}y = 0$

Example 16.18

Find all solutions to

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x$$

Example 16.19

Find all solutions to $2x\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 7x^4$

Example 16.20

Find the initial value solution of $y' + y\tan x = \cos^2 x$, $y(0) = 1$

Example 16.21

Find the solution to:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2t}{1+t^2}y + \frac{2}{1+t^2}, \qquad y(0) = 0$$

Example 16.22

Solve the following initial value problem for $x > 0$

$$2x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 3x + 1, \qquad y(3) = 4$$

Example 16.23

A tank contains 50 litres of a solution composed of 90% water and 10% hydrochloric acid. A second solution containing 50% water and 50% hydrochloric acid is being poured into the tank at a rate of 4 litres per minute. At the same time, the tank is being drained at a rate of 5 litres per minute.

(a) How much acid will there be in the tank after t minutes?

(b) How much acid will there be in the tank after 10 minutes?

Example 16.24

For the differential equation of $\frac{dy}{dx} = x + y$, such that $y(2) = 0$, use Euler's method with a step value of 0.2 to find an approximate value of y when $x = 3$, giving your answer to 2 decimal places.

Example 16.25

Given that $\frac{dy}{dx} = \frac{x+1}{xy+2}$ and $y = 1$ when $x = 0$, use Euler's method with Step size $h = 0.25$ to approximate the value of y when $x = 1$. Give the approximation to 3 significant figures.

Example 16.26

(a) Show that $\sin\theta \approx \theta$ if θ is close to 0 (θ is in radians).

(b) Use the approximation from (a) to approximate $\sin 2°$, and compare your approximation to the result you get from your GDC.

Example 16.27

Find the local linear and quadratic approximations of e at $x = 0$, and graph e^x and the two approximations together.

Example 16.28

Find the Maclaurin polynomials P_0, P_1, P_2, P_3, and P_n, for e^x and graph the first four together.

Example 16.29

Find the Maclaurin series for:

(a) $f(x) = \sin x$ (b) $f(x) = \cos x$ (c) $f(x) = e^x$

Example 16.30

Given that $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, find the Maclaurin series for

$f(x) = \cos x$

Example 16.31

Given that $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, find the Maclaurin series for $f(x) = \sin x$

Example 16.32

Find the Maclaurin series for $f(x) = (1+x)^k$, where k is a real number.

Example 16.33

Find the Maclaurin series for $f(x) = \frac{1}{1+x}$

Example 16.34

Find the Maclaurin series for $g(x) = \dfrac{1}{1-x}$

Example 16.35

Find the Maclaurin series for $g(x) = \dfrac{1}{\sqrt{9-x}}$

Example 16.36

Find the Maclaurin series for $g(x) = \ln(1+x)$

Example 16.37

Find the Maclaurin series for $g(x) = \arctan x$

Example 16.38

(a) Find the Maclaurin series for $f(x) = e^{x^2}$

(b) Hence, find a series for $\int e^{x^2} \mathrm{d}x$

(c) Use the first four terms of the series in (b) to approximate the value of

$$\int_0^1 e^{x^2} \mathrm{d}x.$$

Example 16.39

Find the first three non-zero terms in the Maclaurin series for:

(a) $e^{x^2} \sin x$

(b) $\dfrac{\sin x}{2e^{x^2}}$

Example 16.40

Use the first three terms of an appropriate power series to estimate $\sqrt[5]{33}$

Example 16.41

Use the power series method to solve the differential equation $\frac{dy}{dx} - y = 0$

Example 16.42

Use the power series method to solve the differential equation $y' - xy = 0, y(0) = 2$